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Vertex operators for AdS_3 background with Ramond-Ramond flux

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Abstract: In order to study vertex operators for the Type IIB superstring on AdS space, we derive supersymmetric constraint equations for the vertex operators in $AdS_3 \times S^3$ backgrounds with Ramond-Ramond flux, using Berkovits-Vafa-Witten variables. These constraints are solved to compute the vertex operators and show that they satisfy the linearized $D = 6$, $N = (2, 0)$ equations of motion for a supergravity and tensor multiplet expanded around the $AdS_3 \times S^3$ spacetime.

Keywords: Superstring Vacua, Conformal Field Models in String Theory.

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Contents

1. Introduction

The formulation of compactified string theories on anti de Sitter (AdS) backgrounds is necessary to understand the conjectured dualities with spacetime conformal field theories (CFT's). Presently, calculations using the correspondence are restricted to the supergravity limit of the string theories. This is because of the appearance of background Ramond-Ramond fields in the string worldsheet action, which makes understanding the worldsheet CFT's difficult.

This problem has been partly overcome in some special cases. For example, the Berkovits-Vafa formalism for supersymmetric quantization on $\mathbb{R}^6 \times M$ (where M is $K3$ or $\mathbb{T}^4$) [1, 2, 3] uses the following worldsheet fields. In addition to bosonic fields $x^p(z, \overline{z})$ that contain both left- and right-moving modes, there are left-moving fermi fields $\theta^a_L(z), p^a_L(z)$ of spins 0 and 1, together with ghosts $\sigma_L(z), \rho_L(z)$, and right-moving counterparts of all these left-moving fields. The virtue of this formalism is that Ramond-Ramond background fields can be incorporated without adding spin fields to the worldsheet action. In the $AdS_3 \times S^3$ case, after adding those backgrounds it becomes convenient [4] to integrate out the p's, giving a model in which the fields are x^p, θ^a, $\overline{\theta}^a$ (all now with both left- and right-moving components) as well as the ghosts. x^p, θ^a, and $\overline{\theta}^a$ can be combined together as coordinates on a supergroup manifold PSU(2|2).[1] The model is thus a sigma model with that supergroup as a target, coupled also to the ghosts ρ and σ. The spacetime supersymmetry group is PSU(2|2) $\times$ PSU(2|2), acting by left and right multiplication on the PSU(2|2) manifold, that is, by $g \to agb^{-1}$ where g is a PSU(2|2)-valued field (which combines x, θ, and $\overline{\theta}$), and $a, b \in$ PSU(2|2) are the symmetry group elements.

Thus one arrives [4] at a sigma model with conventional local interactions (no spin fields in the lagrangean) that gives a conformal field theory description of strings propagating on $AdS_3 \times S^3$, with manifest spacetime supersymmetry. This gives a framework for studying the type IIB superstring on $AdS_3 \times S^3 \times M$, and the model can in principle be used to go beyond the supergravity approximation and compute the α' expansion of correlation functions.

In this paper, we analyze the vertex operators for this model for the massless states that are independent of the details of the compactification (including the supergravity multiplet). We work to leading order in α', but because of the high degree of symmetry of the model, it seems very likely that the result is exact. Concretely, we look for vertex operators that are constructed from a function of the group-valued fields g together with the ghosts ρ, σ (but with no dependence on derivatives of the fields). The motivation for this assumption is that it holds in the flat case, according to [1].

The discussion is organized as follows. In section 2, we write the string constraints and find an explicit description of the action of the PSU(2|2)^2 generators on the vertex operators. We construct the constraint equations for the vertex operators in terms of these charges, by requiring the equations to be a smooth deformation of the known flat space case and to be invariant under the AdS supersymmetry transformations. In section 3, we discuss how the gauge transformations of the vertex operator equations distinguish the physical degrees of freedom. In section 4 we evaluate the constraints, using the differential geometry of the group manifold. In section 5, we compute the linearized AdS supergravity equations and match the vertex operator field components to the supergravity fields of general relativity.

2. String constraints

Preliminaries on PSU(2|2) The bosonic part of the Lie algebra of PSU(2|2) is the SU(2) $\times$ SU(2) or SO(4) Lie algebra.[2] The generators $t^{ab} = -t^{ba}$ of SU(2) $\times$ SU(2) transform as an antisymmetric rank two tensor of SU(2) $\times$ SU(2) = SO(4). (Here

[1] PSU(2|2) is the projectivization of SU(2|2), obtained by dividing by its center, which is U(1). PSU(2|2) has been called SU$'$(2|2) in [4].

[2] For $AdS_3 \times S^3$, we really want the real form to be SU(2) $\times$ SL(2, $\mathbb{R}$), but for our formal discussion we will just take the real form to be SU(2) $\times$ SU(2).

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$a, b = 1, \ldots, 4$ are vector indices of SO(4).) The fermionic generators e^a and f^a transform as vectors of SO(4). The Lie superalgebra takes the form

$$
\begin{aligned}
[t_{ab}, t_{cd}] &= \delta_{ac}\, t_{bd} + \delta_{bd}\, t_{ac} - \delta_{ad}\, t_{bc} - \delta_{bc}\, t_{ad}\,, \\
[t_{ab},\, f_c] &= \delta_{ac} f_b - \delta_{bc} f_a\,, \\
[t_{ab},\, e_c] &= \delta_{ac} e_b - \delta_{bc} e_a\,, \\
\{e_a,\, f_b\} &= \frac{1}{2}\epsilon_{abcd}\, t^{cd}\,, \\
\{e_a, e_b\} &= \{f_a, f_b\} = 0\,.
\end{aligned}
\tag{2.1}
$$

On the PSU(2|2) manifold, we introduce bosonic and fermionic coordinates $x^p, p = 1, \ldots, 6$ and θ^a, $\overline{\theta}^a$, $a = 1, \ldots, 4$. We write a generic PSU(2|2) element as

$$
g = \exp(\theta^a f_a) h \exp\left(\overline{\theta}^{\overline{b}} e_{\overline{b}}\right), \tag{2.2}
$$

with $h \in \mathrm{SU}(2) \times \mathrm{SU}(2)$. When we want an explicit parametrization for h, we write

$$
h = \exp\left(\frac{1}{2}\sigma^{pcd} x_p t_{cd}\right), \tag{2.3}
$$

where the σ^{pcd} are described in appendix A. The idea here is that in the almost flat limit in which $AdS_3 \times S^3$ reduces to Minkowski space $\mathbb{R}^6$, the x^p become standard Minkowski coordinates. In this limit, the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is extended to $\mathrm{SO}(6) = \mathrm{SU}(4)$, with the p index of x^p transforming as a vector of SO(6), and indices a, b, c transforming as a positive chirality spinor of SO(6) ($\overline{a}$, $\overline{b}$, $\overline{c}$ transform as an SO(6) spinor of the same chirality since we compactify the Type IIB superstring on $AdS_3 \times \mathbf{S}^3 \times M$). We will write E_a, F_a, and K_{ab} for the operators that represent the left action of e_a, f_a, and t_{ab} on g. Very concretely, in the above coordinates,

$$
\begin{aligned}
F_a &= \frac{d}{d\theta^a}\,, \qquad K_{ab} = -\theta_a \frac{d}{d\theta^b} + \theta_b \frac{d}{d\theta^a} + t_{Lab} \\
E_a &= \frac{1}{2}\epsilon_{abcd}\, \theta^b \left(t_L^{cd} - \theta^c \frac{d}{d\theta_d}\right) + h_{a\overline{b}} \frac{d}{d\overline{\theta}_{\overline{b}}}\,.
\end{aligned}
\tag{2.4}
$$

These formulas are found by asking for $F_a g = f_a g$, $E_a g = e_a g$, $K_{ab} g = -t_{ab} g$. Also, we have introduced an operator t_L that generates the left action of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on h alone, without acting on the θ's. We will also use an analogous t_R that acts on h alone, on the right. Explicitly

$$
t_{Lab}\, g = e^{\theta^a f_a}(-t_{ab}) h(x) e^{\overline{\theta}^{\overline{a}} e_{\overline{a}}}\,, \qquad t_{R\overline{a}\overline{b}} g = e^{\theta^a f_a} h(x) t_{\overline{a}\overline{b}}\, e^{\overline{\theta}^{\overline{a}} e_{\overline{a}}} \tag{2.5}
$$

where

$$
g = g(x, \theta\,, \overline{\theta}) = e^{\theta^a f_a}\, e^{\frac{1}{2}\sigma^{pcd} x_p t_{cd}}\, e^{\overline{\theta}^{\overline{a}} e_{\overline{a}}} = e^{\theta^a f_a}\, h(x)\, e^{\overline{\theta}^{\overline{a}} e_{\overline{a}}} \tag{2.6}
$$

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In other words, in the parametrization of PSU(2|2) in terms of h, θ, and $\overline{\theta}$, t_L and t_R generate the left and right $\mathrm{SU}(2)\times\mathrm{SU}(2)$ action on h, leaving θ and $\overline{\theta}$ fixed.

The use of barred and unbarred indices in the above formulas is to be understood as follows. Like any group manifold, the SO(4) manifold admits a left-invariant vielbein and also a right-invariant vielbein. If we write m, n, p for a tangent space index, A, B, C for a local Lorentz index in the right-invariant vielbein, and $\overline{A}, \overline{B}, \overline{C}$ for a local Lorentz index in the left-invariant vielbein, then one might write the right- and left-invariant vielbeins as e_m^A and $\overline{e}_m^{\overline{A}}$, respectively. However, because at every point on the SO(4) manifold, there is a distinguished SO(4) subgroup of the tangent space group SO(6) (namely the subgroup that leaves fixed the given point), we use an SO(4) notation: we write a single local Lorentz index A in the vector of SO(6) (or second rank antisymmetric tensor of SO(4)) as ab (with antisymmetry in a and b understood). Likewise, $\overline{A}$ is replaced by $\overline{a}\overline{b}$. The right- and left-invariant vielbeins are then written as $-\sigma_m^{ab}$ and $\sigma_m^{\overline{a}\overline{b}}$, respectively. SO(4) indices can be raised or lowered as always with the invariant metric tensor δ_{ab} of SO(4). In short, indices a, b, c are local Lorentz indices for the right-invariant framing transforming as spinors of SO(6), or vectors of SO(4); and $\overline{a}, \overline{b}, \overline{c}$ are the same for the left-invariant framing. In (2.4), $h_{a\overline{b}}$ is the SO(4) group element in the vector representation. It carries one index of each kind. It is orthogonal – that is, $h_{a\overline{a}} = h^{-1}_{\overline{a}a}$ – and is acted on very simply by t_L and t_R:

$$h^{a\overline{a}}\, t_L^{cd}\, h^{-1}_{\overline{a}b} = -\delta^{ac}\,\delta^d_b + \delta^{ad}\,\delta^c_b\,, \qquad h^{-1}_{\overline{a}a}\, t_R^{\overline{c}\overline{d}}\, h^{a\overline{b}} = -\delta^{\overline{c}}_{\overline{a}}\,\delta^{\overline{d}\overline{b}} + \delta^{\overline{d}}_{\overline{a}}\,\delta^{\overline{c}\overline{b}}\,. \tag{2.7}$$

We call the generators of the right action of PSU(2|2) on itself $\overline{K}_{\overline{a}\overline{b}}$, $\overline{E}_{\overline{a}}$, and $\overline{F}_{\overline{a}}$. Just as above, we compute

$$\begin{aligned}\overline{E}_{\overline{a}} &= \frac{d}{d\overline{\theta}^{\overline{a}}}\,, \qquad \overline{K}_{\overline{a}\overline{b}} = -\overline{\theta}_{\overline{a}}\frac{d}{d\overline{\theta}^{\overline{b}}} + \overline{\theta}_b\frac{d}{d\overline{\theta}^{\overline{a}}} + t_{R\overline{a}\overline{b}} \\ \overline{F}_{\overline{a}} &= \frac{1}{2}\epsilon_{\overline{a}\overline{b}\overline{c}\overline{d}}\,\overline{\theta}^{\overline{b}}\left(-t_R^{\overline{c}\overline{d}} + \theta^{\overline{c}}\frac{d}{d\overline{\theta}_{\overline{d}}}\right) + h^{-1}_{\overline{a}b}\frac{d}{d\theta_b}\end{aligned} \tag{2.8}$$

from

$$\overline{F}_{\overline{a}}\, g = g\, f_{\overline{a}}\,, \qquad \overline{E}_{\overline{a}}\, g = g\, e_{\overline{a}}\,, \qquad \overline{K}_{\overline{a}\overline{b}}\, g = g\, t_{\overline{a}\overline{b}}\,. \tag{2.9}$$

Indices with bars are related to those without bars through the SO(4) group element $h_{a\overline{a}}$ by

$$G_a = h_a^{\ \overline{b}}\, G_{\overline{b}}, \quad G_{\overline{a}} = h^{-1\,b}_{\overline{a}}\, G_b = h^b_{\ \overline{a}} G_b\,. \tag{2.10}$$

One of the most important objects in PSU(2|2) group theory is the quadratic Casimir operator $f_a e_a + \frac{1}{8}\epsilon_{abcd}t^{ab}t^{cd}$. As an operator on the PSU(2|2) manifold, the quadratic Casimir is a second order differential operator (since the group generators are first order operators); we will call it the laplacean. The laplacean can be written

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in terms of either left or right generators as

$$\begin{aligned} F_a E_a + \frac{1}{8}\epsilon_{abcd} K^{ab} K^{cd} &= \overline{F}_{\overline{a}} \overline{E}_{\overline{a}} + \frac{1}{8}\epsilon_{\overline{a}\overline{b}\overline{c}\overline{d}} \overline{K}^{\overline{a}\overline{b}}\, K^{\overline{c}\overline{d}} \\ &= h_{a\overline{b}} \frac{d}{d\theta_a}\frac{d}{d\overline{\theta}_{\overline{b}}} + \frac{1}{8}\epsilon_{abcd} t_L^{ab} t_L^{cd} = h^{-1}_{\overline{b}a} \frac{d}{d\theta_a}\frac{d}{d\overline{\theta}_{\overline{b}}} + \frac{1}{8}\epsilon_{\overline{a}\overline{b}\overline{c}\overline{d}} \overline{t}_R^{\overline{a}\overline{b}} t_R^{\overline{c}\overline{d}} . \end{aligned} \tag{2.11}$$

In verifying these formulas, one uses the fact that the SO(4) laplacean can similarly be written in terms of left or right generators: $\frac{1}{8}\epsilon_{abcd}\, t_L^{ab}\, t_L^{cd} = \frac{1}{8}\overline{\epsilon}_{\overline{a}\overline{b}\overline{c}\overline{d}}\, \overline{t}_R^{\overline{a}\overline{b}}\, t_R^{\overline{c}\overline{d}}$. It follows from the action of the generators given in (2.8) that $\overline{F}_{\overline{a}}, \overline{E}_{\overline{a}}, \overline{K}_{\overline{a}\overline{b}}$ also obey the PSU(2|2) commutation relations and that the laplacean (2.11) commutes with the group generators (2.4) and (2.8).

We recall finally that PSU(2|2) has an SL(2, $\mathbb{R}$) group of outer automorphisms with E, F transforming as a doublet. In particular, the substitution

$$\overline{F}'_{\overline{a}} = \overline{E}_{\overline{a}}\,, \qquad \overline{E}'_{\overline{a}} = -\overline{F}_{\overline{a}}\,, \qquad \overline{K}'_{\overline{a}\overline{b}} = \overline{K}_{\overline{a}\overline{b}} \tag{2.12}$$

leaves the laplacean and the commutation relations invariant.

Form of vertex operators We now want to construct the vertex operators for the supergravity multiplet. In this, we are guided by comparison with the flat space case [1, 4]. In the flat case, the supergravity vertex operators are constructed by acting with some $N = 4$ generators on a spin zero field $V(x, \theta, \overline{\theta}; \rho + i\sigma, \overline{\rho} + i\overline{\sigma})$ that is a function of these fields but not their derivatives. V also obeys certain additional constraints.

The spin zero property is ensured by having V annihilated by the laplacean $\partial^p \partial_p$. The laplacean is a second order differential operator that is invariant under the Poincaré symmetries. The analogous invariant second order operator on the PSU(2|2) manifold is the quadratic Casimir operator that was introduced above.

The ghost fields ρ and σ enter only in the combinations $\rho + i\sigma$ and its barred counterpart because these combinations have zero background charge and so do not contribute to the dimension of the operator. Also, these are the combinations that remain holomorphic after perturbing to $AdS_3 \times S^3$ as in [4].

Finally, the constraint equations in the flat space case, as presented in [1, 4], are written in terms of $\partial^{ab} = \sigma^{pab}\partial/\partial x^p$ and $\nabla_a = \partial/\partial\theta^a$. In the curved case, ∂^{ab} is deformed to K^{ab} and ∇_a is interpreted as F_a. To the flat space constraint equations, we find that we have to add lower order terms to achieve PSU(2|2) invariance.

The expansion of the vertex operator in terms of the ghost fields is

$$V = \sum_{m,n=-\infty}^{\infty} e^{m(i\sigma+\rho)+n(i\overline{\sigma}+\overline{\rho})} V_{m,n}(x, \theta, \overline{\theta})\,. \tag{2.13}$$

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In flat space, the constraints from the left and right-moving worldsheet super Virasoro algebras are from [4]:

$$
\begin{aligned}
(\nabla)^4 V_{1,n} &= \nabla_a \nabla_b \partial^{ab} V_{1,n} = 0\,, \\
\frac{1}{6}\epsilon^{abcd}\,\nabla_b\,\nabla_c\,\nabla_d V_{1,n} &= -i\nabla_b\,\partial^{ab} V_{0,n}\,, \\
\nabla_a\,\nabla_b\,V_{0,n} - \frac{i}{2}\epsilon_{abcd}\,\partial^{cd}\,V_{-1,n} &= 0\,, \\
\nabla_a\,V_{-1,n} &= 0\,;
\end{aligned} \tag{2.14}
$$

$$
\begin{aligned}
\overline{\nabla}^4 V_{n,1} &= \overline{\nabla}_{\overline{a}}\overline{\nabla}_{\overline{b}}\overline{\partial}^{\overline{a}\overline{b}} V_{n,1} = 0\,, \\
\frac{1}{6}\epsilon^{\overline{a}\overline{b}\overline{c}\overline{d}}\overline{\nabla}_{\overline{b}}\overline{\nabla}_{\overline{c}}\overline{\nabla}_{\overline{d}} V_{n,1} &= -i\overline{\nabla}_{\overline{b}}\overline{\partial}^{\overline{a}\overline{b}} V_{n,0}\,, \\
\overline{\nabla}_{\overline{a}}\overline{\nabla}_{\overline{b}} V_{n,0} - \frac{i}{2}\overline{\epsilon}_{\overline{a}\overline{b}\overline{c}\overline{d}}\,\overline{\partial}^{\overline{c}\overline{d}}\,V_{n,-1} &= 0\,, \\
\overline{\nabla}_{\overline{a}}\,V_{n,-1} &= 0\,;
\end{aligned} \tag{2.15}
$$

$$
\partial^p \partial_p V_{m,n} = 0\,, \tag{2.16}
$$

for $-1 \leq m, n \leq 1$, with the notation $\nabla_a = d/d\theta^a$, $\overline{\nabla}_{\overline{a}} = d/d\overline{\theta}^{\overline{a}}$, $\partial^{ab} = -\sigma^{pab}\,\partial_p$. In flat space, these equations were derived by requiring the vertex operators to satisfy the physical state conditions

$$
\begin{aligned}
G_0^- V = \tilde{G}_0^- V = \overline{G}_0^- V = \overline{\tilde{G}}_0^- V = T_0 V = \overline{T}_0 V = 0\,, \\
J_0 V = \overline{J}_0 V = 0\,, \qquad G_0^+ \tilde{G}_0^+ V = \overline{G}_0^+ \overline{\tilde{G}}_0^+ V = 0\,,
\end{aligned} \tag{2.17}
$$

where T_n, $G_n^\pm$, $\tilde{G}_n^\pm$, J_n, $J_n^\pm$ and corresponding barred generators are the left and right $N = 4$ worldsheet superconformal generators present in the Berkovits-Vafa quantization of the type II superstring [1, 2], [4]. The conditions (2.17) further implied $V_{m,n} = 0$ for $m > 1$ or $n > 1$ or $m < 1$ or $n < 1$, leaving nine non-zero components.

In curved space, we modify these equations as follows:

$$
\begin{aligned}
F^4 V_{1,n} &= F_a\,F_b K^{ab} V_{1,n} = 0\,, \\
\frac{1}{6}\epsilon^{abcd}\,F_b\,F_c\,F_d V_{1,n} &= -iF_b\,K^{ab} V_{0,n} + 2iF^a V_{0,n} - E^a V_{-1,n}\,, \\
F_a\,F_b\,V_{0,n} - \frac{i}{2}\epsilon_{abcd}\,K^{cd}\,V_{-1,n} &= 0\,, \\
F_a\,V_{-1,n} &= 0\,;
\end{aligned} \tag{2.18}
$$

$$
\begin{aligned}
\overline{F}^4 V_{n,1} &= \overline{F}_{\overline{a}}\overline{F}_{\overline{b}}\overline{K}^{\overline{a}\overline{b}} V_{n,1} = 0\,, \\
\frac{1}{6}\epsilon^{\overline{a}\overline{b}\overline{c}\overline{d}}\overline{F}_{\overline{b}}\overline{F}_{\overline{c}}\overline{F}_{\overline{d}} V_{n,1} &= -i\overline{F}_{\overline{b}}\overline{K}^{\overline{a}\overline{b}} V_{n,0} + 2i\overline{F}^{\overline{a}} V_{n,0} - \overline{E}^{\overline{a}} V_{n,-1}\,, \\
\overline{F}_{\overline{a}}\overline{F}_{\overline{b}} V_{n,0} - \frac{i}{2}\overline{\epsilon}_{\overline{a}\overline{b}\overline{c}\overline{d}}\,\overline{K}^{\overline{c}\overline{d}}\,V_{n,-1} &= 0\,, \\
\overline{F}_{\overline{a}}\,V_{n,-1} &= 0\,.
\end{aligned} \tag{2.19}
$$

There is also a spin zero condition constructed from the laplacean:

$$\left(F_a\,E_a+\frac{1}{8}\epsilon_{abcd}\,K^{ab}\,K^{cd}\right)\,V_{n,m}=\left(\overline{F}_{\overline{a}}\,\overline{E}_{\overline{a}}+\frac{1}{8}\overline{\epsilon}_{\overline{a}\overline{b}\overline{c}\overline{d}}\overline{K}^{\overline{a}\overline{b}}\,K^{\overline{c}\overline{d}}\right)\,V_{n,m}=0\,. \qquad (2.20)$$

Equations (2.18)–(2.20) have been derived by deforming the corresponding equations for the flat case, which were presented above, by requiring invariance under the PSU(2|2) transformations that we will present momentarily. For example, in the second equation in (2.18), the terms on the right hand side that are linear in PSU(2|2) generators are curvature corrections to the flat equations, while the quadratic and cubic terms are present in the flat case.

The above constraint equations are invariant under the action of PSU(2|2) $\times$ PSU(2|2), but there is a subtlety in how PSU(2|2) acts. The subtlety comes from the unusual form of the PSU(2|2) $\times$ PSU(2|2) currents found in [4] (and ultimately from the unusual form of half of the supercharges even in the flat case in this formalism [1]). While F and K have natural definitions in [4], and act by the obvious left multiplication on g while leaving invariant the ghosts, this is not true for E. Rather, E has a term that acts naturally on g, plus corrections that are proportional to $e^{-\rho-i\sigma}$ times F.

We have found that the following transformations generate a PSU(2|2) action on the V's that commutes with the constraint equations in (2.18)–(2.20):

$$\begin{aligned}\Delta^-_a\,V_{m,n}&=F_a\,V_{m,n}\,,\\ \Delta_{ab}\,V_{m,n}&=K_{ab}\,V_{m,n}\\ \Delta^+_a\,V_{1,n}&=E_a\,V_{1,n}\,,\\ \Delta^+_a\,V_{0,n}&=E_a\,V_{0,n}+iF_aV_{1,n}\,,\\ \Delta^+_a\,V_{-1,n}&=E_a\,V_{-1,n}-iF_aV_{0,n}\,.\end{aligned} \qquad (2.21)$$

(In a hopefully obvious notation, we write Δ_{ab} for the variation of the vertex operator generated by t_{ab} and $\Delta^\pm_a$ for the variations generated by e_a and f_a.) These formulas are the standard PSU(2|2) generators on the $V_{m,n}$, except that to E we have added a multiple of F times a "raising" operator on the m index. This is suggested by the form of the supercharges $q^\pm_a$ in [4]. The constraint equations in (2.18)–(2.20) were determined by being invariant under the deformed PSU(2|2) generators. This uniquely fixed all the "lower order terms" that were not present in the flat case. It would be desirable to derive (2.18)–(2.20) directly by constructing the constraint operators and their action on vertex functions exactly, or at least perturbatively in α'. However, the uniqueness gives us confidence in the formulas.

3. Gauge transformations

The constraint equations for the vertex operators are also invariant under gauge transformations. We use these to identify the physical degrees of freedom of the

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vertex operator (2.13) as follows. Useful gauge symmetries of the flat space equations are

$$\delta V = G_0^+ \Lambda + \tilde{G}_0^+ \tilde{\Lambda} + \overline{G}_0^+ \overline{\Lambda} + \overline{\tilde{G}_0}^+ \overline{\tilde{\Lambda}} , \qquad \delta V = \tilde{G}_0^+ \tilde{G}_0^- \overline{\tilde{G}_0^+ \tilde{G}_0^-} \Omega \tag{3.1}$$

with (sum on $n = -1, 0, 1$)

$$\begin{aligned} \Lambda &= e^{i\sigma + 2\rho + n(i\overline{\sigma} + \overline{\rho})} \lambda_n(x, \theta, \overline{\theta}) & \qquad \overline{\Lambda} &= e^{i\overline{\sigma} + 2\overline{\rho} + n(i\sigma + \rho)} \overline{\lambda}_n(x, \theta, \overline{\theta}) \\ \tilde{\Lambda} &= e^{-\rho - iH_c + n(i\overline{\sigma} + \overline{\rho})} \tilde{\lambda}_n(x, \theta, \overline{\theta}) & \qquad \overline{\tilde{\Lambda}} &= e^{-\overline{\rho} - i\overline{H}_c + n(i\sigma + \rho)} \overline{\tilde{\lambda}}_n(x, \theta, \overline{\theta}) , \end{aligned}$$

where the gauge parameters Λ,$\tilde{\Lambda}$ are annihilated by $\overline{G}_0^+ \overline{\tilde{G}_0}^+$, the gauge parameters $\overline{\Lambda}$,$\overline{\tilde{\Lambda}}$ are annihilated by $G_0^+ \tilde{G}_0^+$, all $\Lambda, \tilde{\Lambda}, \overline{\Lambda}, \overline{\tilde{\Lambda}}$ are annihilated by $T_0, \overline{T}_0, G_0^-, \tilde{G}_0^-, \overline{G}_0^-, \overline{\tilde{G}_0}^-$, and Ω is annihilated by $T_0, \overline{T}_0$. The currents $e^{-iH_c}, e^{-i\overline{H}_c}$ are related to the $N = 4$ currents $J^-(z) = e^{-\rho - i\sigma} e^{-iH_c}$, $\overline{J}^-(\overline{z}) = e^{-\overline{\rho} - i\overline{\sigma}} e^{-i\overline{H}_c}$. Since fermionic conformal fields do not commute, i.e. $e^{i\sigma(z)}\, e^{\rho(\zeta)} \sim -e^{\rho(\zeta)}\, e^{i\sigma(z)}$, the notation is defined as $e^{i\sigma + \rho} \equiv e^{i\sigma}\, e^{\rho}$, etc.

Using the transformations (3.1) one can gauge fix to zero the vertex operators $V_{-1,1}, V_{1,-1}, V_{0,-1}, V_{-1,0}, V_{-1,-1}$, and therefore they do not correspond to propagating degrees of freedom. Furthermore this gauge symmetry can be used both to set to zero the components of $V_{1,1}$ with no θ's or no $\overline{\theta}$'s, and to gauge fix all components of $V_{0,1}, V_{1,0}, V_{0,0}$ that are independent of those of $V_{1,1}$. The physical degrees of freedom are thus described by a superfield [4]

$$\begin{aligned} V_{1,1} = {} & \theta^a \overline{\theta}^{\overline{a}} V_{a\overline{a}}^{--} + \theta^a \theta^b \overline{\theta}^{\overline{a}} \sigma_{ab}^m \overline{\xi}_{m\,\overline{a}}^- + \theta^a \overline{\theta}^{\overline{a}} \overline{\theta}^{\overline{b}} \sigma_{\overline{a}\overline{b}}^m \xi_{m\,a}^- + \\ & + \theta^a \theta^b \overline{\theta}^{\overline{a}} \overline{\theta}^{\overline{b}} \sigma_{ab}^m \sigma_{\overline{a}\overline{b}}^n (g_{mn} + b_{mn} + \overline{g}_{mn} \phi) + \theta^a (\overline{\theta}^3)_{\overline{a}} A_a^{-+\overline{a}} + (\theta^3)_a \overline{\theta}^{\overline{a}} A_{\overline{a}}^{+-\,a} + \\ & + \theta^a \theta^b (\overline{\theta}^3)_{\overline{a}} \sigma_{ab}^m \overline{\chi}_m^{+\,\overline{a}} + (\theta^3)^a \overline{\theta}^{\overline{a}} \overline{\theta}^{\overline{b}} \sigma_{\overline{a}\overline{b}}^m \chi_m^{+\,a} + (\theta^3)_a (\overline{\theta}^3)_{\overline{a}} F^{++\,a\overline{a}} . \end{aligned} \tag{3.2}$$

This has the field content of $D = 6$, $N = (2, 0)$ supergravity with one supergravity and one tensor multiplet [5]. As explained in [4], in flat space, the surviving constraint equations in the set (2.14)–(2.16) imply that the component fields Φ are all on shell massless fields, that is $\sum_{m=1}^6 \partial^m \partial_m \Phi = 0$ and in addition

$$\begin{aligned} \partial^m g_{mn} &= -\partial_n \phi , \\ \partial^m b_{mn} &= 0 , \\ \partial^m \chi_m^{\pm b} = \partial^m \overline{\chi}_m^{\pm \overline{b}} &= 0 , \\ \partial_{ab} \chi_m^{\pm b} &= \partial_{\overline{a}\overline{b}} \chi_m^{\pm \overline{b}} = 0 , \\ \partial_{cb} F^{\pm\pm b\overline{a}} &= \partial_{\overline{c}\overline{b}} F^{\pm\pm \overline{b} a} = 0 , \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} F^{+-a\overline{a}} &= \partial^{\overline{a}\overline{b}} A_{\overline{b}}^{+-a} , & \qquad \chi_m^{-a} &= \partial^{ab} \xi_{mb}^- , \\ F^{-+a\overline{a}} &= \partial^{ab} A_b^{-+\overline{a}} , & \qquad \overline{\chi}_m^{-\overline{a}} &= \partial^{\overline{a}\overline{b}} \overline{\xi}_{m\overline{b}}^- . \\ F^{--a\overline{a}} &= \partial^{ab} \partial^{\overline{a}\overline{b}} V_{b\overline{b}}^{--} \end{aligned}$$

The equations of motion (3.3) for the flat space vertex operator component fields describe $D = 6$, $N = (2,0)$ supergravity expanded around the six-dimensional Minkowski metric.

In $AdS_3 \times S^3$ space there are corresponding gauge transformations which reduce the number of degrees of freedom to those in (3.2), but the laplacean must be replaced by the AdS laplacean, and the constraints are likewise deformed. In section 4, we focus on the vertex operator V_{11} that carries the physical degrees of freedom. We derive the conditions on its field components that follow from the AdS vertex operator constraint equations (2.18)–(2.20) , and give their residual gauge invariances explicitly. In section 5, we show these conditions are equivalent to the $D = 6, N = (2,0)$ linearized supergravity equations expanded around the $AdS_3 \times S^3$ metric.

4. String equations for AdS vertex operator field components

Acting on the superfield in (3.2), the AdS supersymmetric constraints (2.18)–(2.20) imply

$$\begin{aligned} F_a\, F_b\, K^{ab} V_{1,1} &= 0\,, \\ \overline{F}_{\overline{a}} \overline{F}_{\overline{b}}\, \overline{K}^{\overline{a}\overline{b}} V_{1,1} &= 0 \end{aligned} \tag{4.1}$$

$$\left(F_a\, E_a\, + \frac{1}{8} \epsilon_{abcd}\, K^{ab}\, K^{cd} \right) V_{1,1} = \left(\overline{F}_{\overline{a}}\, \overline{E}_{\overline{a}}\, + \frac{1}{8} \epsilon_{\overline{a}\overline{b}\overline{c}\overline{d}}\, \overline{K}^{\overline{a}\overline{b}}\, K^{\overline{c}\overline{d}} \right) V_{1,1} = 0\,. \tag{4.2}$$

For the bosonic field components of the vertex operators, the zero laplacean condition (4.2) requires that

$$\Box\, h^g_{\ \overline{a}}\, V^{--}_{ag} = -4\, \sigma^m_{ab}\, \sigma^n_{gh}\, \delta^{bh}\, h^g_{\ \overline{a}}\, G_{mn} \tag{4.3}$$

$$\Box\, h^g_{\ \overline{a}}\, h^h_{\ \overline{b}}\, \sigma^m_{ab}\, \sigma^n_{gh}\, G_{mn} = \frac{1}{4} \epsilon_{abce} \epsilon_{fghk}\, \delta^{ch}\, h^f_{\ \overline{a}}\, h^g_{\ \overline{b}}\, F^{++ek} \tag{4.4}$$

$$\begin{aligned} \Box\, h_g^{\ \overline{a}}\, F^{++ag} &= 0\,, \\ \Box\, h_g^{\ \overline{a}}\, A_a^{-+g} &= 0\,, \\ \Box\, h^g_{\ \overline{a}}\, A_g^{+-a} &= 0 \end{aligned} \tag{4.5}$$

and (4.1) results in

$$\epsilon_{eacd}\, t_L^{cd}\, h^b_{\ \overline{a}}\, A_b^{+-a} = 0\,, \qquad \epsilon_{\overline{e}\overline{b}\overline{c}\overline{d}}\, t_R^{\overline{c}\overline{d}}\, h_a^{\ \overline{a}}\, A_{\overline{a}}^{-+\overline{b}} = 0\,, \tag{4.6}$$

$$\epsilon_{eacd}\, t_L^{cd}\, h_b^{\ \overline{a}}\, F^{++ab} = 0\,, \qquad \epsilon_{\overline{e}\overline{b}\overline{c}\overline{d}}\, t_R^{\overline{c}\overline{d}}\, h^a_{\ \overline{a}}\, F^{++\overline{a}\overline{b}} = 0\,, \tag{4.7}$$

$$t_L^{ab}\, h^g_{\ \overline{a}}\, h^h_{\ \overline{b}}\, \sigma^m_{ab}\, \sigma^n_{gh}\, G_{mn} = 0\,, \qquad t_R^{\overline{a}\overline{b}}\, h_a^{\ \overline{g}}\, h_b^{\ \overline{h}}\, \sigma^m_{\overline{g}\overline{h}}\, \sigma^n_{\overline{a}\overline{b}}\, G_{mn} = 0\,. \tag{4.8}$$

We have used (2.10) to relate barred and unbarred indices. We have expanded $G_{mn} = g_{mn} + b_{mn} + \overline{g}_{mn}\phi$. The SO(4) laplacean is $\Box \equiv \frac{1}{8}\epsilon_{abcd}\, t_L^{ab}\, t_L^{cd} = \frac{1}{8}\epsilon_{\overline{a}\overline{b}\overline{c}\overline{d}}\, t_R^{\overline{a}\overline{b}}\, t_R^{\overline{c}\overline{d}}$. From identities such as $\frac{1}{2}\epsilon_{deab} \left(t_L^{cd}\, t_L^{ab} + t_L^{ab}\, t_L^{cd} \right) = -\frac{1}{4}\, \delta^c_e\, \epsilon_{abfg}\, t_L^{ab}\, t_L^{fg}$ we find that (4.5) follows from (4.6) and (4.7).

In order to compare this with supergravity, we want to reexpress the above formulas in terms of covariant derivatives D_p on the group manifold. We will, however, write everything in terms of right- or left-invariant vielbeins described in section 2 and appendix B. So we write

$$\mathcal{T}_L^{cd} \equiv -\sigma^{p\,cd}\,D_p\,, \qquad \mathcal{T}_R^{\overline{c}\overline{d}} \equiv \sigma^{p\,\overline{c}\overline{d}}\,D_p\,. \tag{4.9}$$

We also have corresponding objects $t_L^{cd} = -\sigma^{p\,cd}D'_p$, $t_R^{cd} = \sigma^{p\,\overline{c}\overline{d}}D''_p$, where (in contrast to D_p which is the covariant derivative with the Levi-Civita connection) D' and D'' are covariant derivatives defined such that right- or left-invariant vector fields are covariantly constant. Acting on a function, $\mathcal{T}_L = t_L$ and $\mathcal{T}_R = t_R$, since both just act geometrically. But they differ in acting on fields that carry spinor or vector indices. For example, on spinor indices,

$$t_L^{ab}V_e = \mathcal{T}_L^{ab}\,V_e + \frac{1}{2}\delta_e^a\,\delta^{bc}V_c - \frac{1}{2}\delta_e^b\,\delta^{ac}V_c \tag{4.10}$$

so that the invariant derivatives on the SO(4) group manifold satisfy

$$\begin{aligned} t_L^{cd}\sigma^n_{ef}f_n(x) &\equiv \mathcal{T}_L^{cd}\,\sigma^n_{ef}\,f_n(x) + \frac{1}{2}f_{ef}^{\;\;ghcd}\,\sigma^n_{gh}f_n(x)\,, \\ t_R^{\overline{c}\overline{d}}\,\sigma^n_{\overline{e}\overline{f}}\,f_n(x) &\equiv \mathcal{T}_R^{\overline{c}\overline{d}}\,\sigma^n_{\overline{e}\overline{f}}f_n(x) - \frac{1}{2}f^{\overline{c}\overline{d}\overline{g}\overline{h}}{}_{\overline{e}\overline{f}}\sigma^n_{\overline{g}\overline{h}}f_n(x)\,, \end{aligned} \tag{4.11}$$

where the SO(4) structure constants from (2.1) are

$$f_{ef}^{\;\;ghcd} = \frac{1}{2}\Big[\delta^{gc}\delta_e^h\delta_f^d - \delta^{gd}\delta_e^h\delta_f^c - \delta^{hc}\delta_e^g\delta_f^d + \delta^{hd}\delta_e^g\delta_f^c - (e\leftrightarrow f)\Big]\,. \tag{4.12}$$

These definitions of the invariant derivatives are, for example, compatible with

$$\frac{1}{8}\epsilon_{abcd}\,t_L^{ab}\,t_L^{cd}\,\sigma^n_{ef}\,f_n(x) = \frac{1}{8}\epsilon_{\overline{a}\overline{b}\overline{c}\overline{d}}\,t_R^{\overline{a}\overline{b}}\,t_R^{\overline{c}\overline{d}}\,\sigma^n_{ef}\,f_n(x)\,. \tag{4.13}$$

For a more detailed discussion, see appendix B.

4.1 Gauge conditions

We show that the constraints on the vertex operators (4.3)–(4.8) are identical to those of AdS supergravity as follows. Using

$$\begin{aligned} t_L^{ab}\,h^g_{\;\overline{a}}\,h^h_{\;\overline{b}}\,\sigma^m_{ab}\,\sigma^n_{gh}\,G_{mn} = {}& (t_L^{ab}\,h^g_{\;\overline{a}})h^h_{\;\overline{b}}\,\sigma^m_{ab}\,\sigma^n_{gh}\,G_{mn} + \\ & + h^g_{\;\overline{a}}\,(t_L^{ab}\,h^h_{\;\overline{b}})\sigma^m_{ab}\,\sigma^n_{gh}\,G_{mn} + h^g_{\;\overline{a}}\,h^h_{\;\overline{b}}\,(t_L^{ab}\,\sigma^m_{ab}\,\sigma^n_{gh}\,G_{mn}) \end{aligned}$$

and (2.7), we find that (4.8) are gauge conditions on the string fields:

$$D^p g_{ps} = -D_s\phi - \frac{1}{2}(\sigma^m\sigma_s\sigma^n)_{ab}\,\delta^{ab}\,b_{mn} = D^p g_{sp}\,, \qquad D^p b_{ps} = 0 = D^p b_{sp}\,. \tag{4.14}$$

This is the curved space analog of the flat space conditions $\partial^p g_{ps} = -\partial_s\phi$, $\partial^p b_{ps} = 0$.

The string equations (4.3)–(4.8) are invariant under the residual gauge transformations which transform G_{mn}:

$$\begin{aligned}
\triangle g_{ps} &= \frac{1}{2}(D_p\xi_s + D_s\xi_p)\,, \\
\triangle b_{ps} &= \frac{1}{2}(D_p\eta_s - D_s\eta_p) + \frac{1}{2}(\sigma_p\sigma_s\sigma_q)_{ab}\delta^{ab}\xi^q\,, \\
\triangle\phi &= 0\,, \\
\triangle V_{ag}^{--} &= \frac{1}{2}\epsilon_{abcd}\, t_L^{cd}\Lambda^b_{\ g} + \frac{1}{2}h_g^{\ \overline{a}}\epsilon_{\overline{a}\overline{b}\overline{c}\overline{d}}\, t_R^{\overline{c}\overline{d}}\overline{\Lambda}_a^{\ \overline{b}}\,,
\end{aligned} \tag{4.15}$$

where the gauge parameters ξ_m, η_m satisfy

$$\begin{aligned}
D^pD_p\xi_m + (\sigma^p\sigma_m\sigma^n)_{ab}\delta^{ab}D_p\eta_n + \overline{R}_{mp}\xi^p &= 0\,, \\
D^pD_p\eta_m + (\sigma^p\sigma_m\sigma^n)_{ab}\delta^{ab}D_p\xi_n + \overline{R}_{mp}\eta^p &= 0\,, \\
D^m\xi_m = D^m\eta_m &= 0\,,
\end{aligned}$$

V_{ag}^{--} transforms with gauge parameters

$$\Box\Lambda^b_{\ \overline{a}} = 4h^{b\overline{b}}\sigma^m_{\overline{b}\overline{a}}(\xi_m + \eta_m)\,, \qquad \Box\overline{\Lambda}_a^{\ \overline{b}} = -4h^{b\overline{b}}\sigma^m_{ab}(\xi_m - \eta_m)$$

and the $AdS_3 \times S^3$ Ricci tensor $\overline{R}_{rp}$ is

$$\overline{R}_{rp} \equiv -\frac{1}{2}\,\sigma_r^{ab}\sigma_p^{cd}\delta_{ac}\delta_{bd}\,. \tag{4.16}$$

The other string fields are invariant. There are additional independent gauge symmetries which transform $A_{\overline{a}}^{+-b}, A_a^{-+\overline{b}}$, but leave invariant $F^{+-a\overline{b}}, F^{-+a\overline{b}}$ defined in section 4.3. For $AdS_3 \times S^3$ we can write the Riemann tensor and the metric tensor as

$$\begin{aligned}
\overline{R}_{mnp\tau} &= \frac{1}{4}\left(\overline{g}_{m\tau}\overline{R}_{np} + \overline{g}_{np}\overline{R}_{m\tau} - \overline{g}_{n\tau}\overline{R}_{mp} - \overline{g}_{mp}\overline{R}_{n\tau}\right)\,, \\
\overline{g}_{mn} &= \frac{1}{2}\,\sigma_m^{ab}\,\sigma_{n\,ab}\,.
\end{aligned} \tag{4.17}$$

4.2 Metric, dilaton, and two-form

We find from the string constraint (4.4) that the six-dimensional metric field g_{rs}, the dilaton ϕ, and the two-form b_{rs} satisfy

$$\begin{aligned}
\frac{1}{2}D^pD_pb_{rs} = &-\frac{1}{2}(\sigma_r\sigma^p\sigma^q)_{ab}\delta^{ab}\,D_p\left[\,g_{qs} + \overline{g}_{qs}\phi\,\right] + \frac{1}{2}(\sigma_s\sigma^p\sigma^q)_{ab}\delta^{ab}D_p\left[\,g_{qr} + \overline{g}_{qr}\phi\,\right] - \\
&- \overline{R}_{\tau rs\lambda}\,b^{\tau\lambda} - \frac{1}{2}\overline{R}_r^{\ \tau}\,b_{\tau s} - \frac{1}{2}\overline{R}_s^{\ \tau}\,b_{r\tau} + \frac{1}{4}F_{\text{asy}}^{++gh}\,\sigma_r^{ab}\sigma_s^{ef}\,\delta_{ah}\delta_{be}\delta_{gf}
\end{aligned} \tag{4.18}$$

and

$$\frac{1}{2}D^pD_p\,(\,g_{rs}+\overline{g}_{rs}\phi\,)\;=\;-\frac{1}{2}(\sigma_r\sigma^p\sigma^q)_{ab}\delta^{ab}\,D_pb_{qs}\;+\frac{1}{2}(\sigma_s\sigma^p\sigma^q)_{ab}\delta^{ab}\,D_pb_{rq}\,-$$
$$-\,\overline{R}_{\tau rs\lambda}\,(\,g^{\tau\lambda}+\overline{g}^{\tau\lambda}\phi\,)\;-\frac{1}{2}\overline{R}_r^{\ \tau}\,(\,g_{\tau s}+\overline{g}_{\tau s}\phi\,)\,-$$
$$-\,\frac{1}{2}\overline{R}_s^{\ \tau}\,(\,g_{r\tau}+\overline{g}_{r\tau}\phi\,)+\frac{1}{4}F^{++gh}_{\rm sym}\,\sigma_{rga}\sigma_{shb}\,\delta^{ab}\,. \tag{4.19}$$

This is the curved space version of the flat space zero laplacean condition $\partial^p\partial_pb_{rs}=\partial^p\partial_pg_{rs}=\partial^p\partial_p\phi=0$ from (3.3).

4.3 Self-dual tensors and scalars

Four self-dual tensor and scalar pairs come from the string bispinor fields F^{++ab}, V^{--}_{ab}, A_b^{+-a}, A_a^{-+b}. From (4.7), we have

$$\sigma^p_{da}\,D_p\,F^{++ab}_{\rm asy}\;=0 \tag{4.20}$$

$$\frac{1}{4}[\delta^{Ba}\,\sigma^r_{ga}\,D_r\,F^{++gH}_{\rm sym}-\delta^{Ha}\,\sigma^r_{ga}\,D_r\,F^{++gB}_{\rm sym}]\;=\;-\frac{1}{4}\,\epsilon^{BH}{}_{cd}\,F^{++cd}_{\rm asy}\,. \tag{4.21}$$

From (4.3), we find

$$\frac{1}{2}\,D^pD_p\,V^{--}_{cd}\;-\frac{1}{2}\delta^{gh}\sigma^p_{ch}\,D_p\,V^{--}_{gd}\;+\frac{1}{2}\delta^{gh}\sigma^p_{dh}\,D_p\,V^{--}_{cg}\;+\frac{1}{4}\epsilon_{cd}{}^{gh}\,V^{--}_{gh}\;-4\,\sigma^m_{ce}\,\sigma^n_{df}\,\delta^{ef}\,G_{mn}\,. \tag{4.22}$$

The string constraints (4.6) can be written as

$$\epsilon_{eacd}\,t^{cd}_L\,h_b^{\ \overline{a}}\,F^{+-ab}=0\qquad\quad \epsilon_{\overline{e}\overline{b}\overline{c}\overline{d}}\,t^{\overline{c}\overline{d}}_R\,h^{\ a}_{\overline{a}}\,F^{+-\overline{a}\overline{b}}=0$$
$$\epsilon_{eacd}\,t^{cd}_L\,h_b^{\ \overline{a}}\,F^{-+ab}=0\qquad\quad \epsilon_{\overline{e}\overline{b}\overline{c}\overline{d}}\,t^{\overline{c}\overline{d}}_R\,h^{\ a}_{\overline{a}}\,F^{-+\overline{a}\overline{b}}=0\,, \tag{4.23}$$

where

$$F^{+-a\overline{a}}\equiv\delta^{\overline{a}\overline{b}}\,A_{\overline{b}}^{+-a}+t^{\overline{a}\overline{b}}_R\,A_{\overline{b}}^{+-a}$$
$$F^{-+a\overline{a}}\equiv\delta^{ab}\,A_b^{-+\overline{a}}+t^{ab}_L\,A_b^{-+\overline{a}}\,,$$

which is similar to the form of (4.7), so F^{+-ab} and F^{-+ab} also satisfy equations of the form (4.20), (4.21).

4.4 Gravitinos and spinors

Independent conditions on the fermion fields from (4.1), (4.2) are

$$\Box\,h_a^{\ \overline{g}}\,\sigma^m_{\overline{a}\overline{b}}\,\xi^-_{m\overline{g}}\;=\;-\sigma^m_{\overline{g}\overline{h}}\,\epsilon_{\overline{e}\overline{d}\overline{a}\overline{b}}\,h_a^{\ \overline{h}}\,\delta^{\overline{g}\overline{d}}\,\overline{\chi}^{+\overline{e}}_m$$

$$\Box\,h^g_{\ \overline{a}}\,\sigma^m_{ab}\,\overline{\xi}^-_{mg}\;=\;-\sigma^m_{gh}\,\epsilon_{edab}\,h^h_{\ \overline{a}}\,\delta^{gd}\,\chi^{+e}_m$$

$$t^{ab}_L\,h^g_{\ \overline{a}}\,\sigma^m_{ab}\,\overline{\xi}^-_{mg}\;=0\,,\qquad\qquad t^{\overline{a}\overline{b}}_R\,h_a^{\ \overline{g}}\,\sigma^m_{\overline{a}\overline{b}}\,\xi^-_{m\overline{g}}\;=0$$

$$t^{ab}_L\,\sigma^m_{ab}\,h_g^{\ \overline{a}}\,\overline{\chi}^{+g}_m\;=0\,,\qquad\qquad t^{\overline{a}\overline{b}}_R\,\sigma^m_{\overline{a}\overline{b}}\,h^{\ a}_{\overline{g}}\,\chi^{+\overline{g}}_m\;=0$$

$$\epsilon_{deab}\,t^{ab}_L\,h^g_{\ \overline{a}}\,h^h_{\ \overline{b}}\,\sigma^m_{gh}\,\chi^{+e}_m\;=0\,,\qquad \epsilon_{\overline{d}\overline{e}\overline{a}\overline{b}}\,t^{\overline{a}\overline{b}}_R\,h_a^{\ \overline{g}}\,h_b^{\ \overline{h}}\,\sigma^m_{\overline{g}\overline{h}}\overline{\chi}^{+\overline{e}}_m\;=0\,. \tag{4.24}$$

5. $D = 6$, $N = (2, 0)$ supergravity on $AdS_3 \times S^3$

In this section we show that the string constraints described in section 5, which were derived from the $AdS_3 \times S^3$ supersymmetric vertex operator equations (2.18)–(2.20), are equivalent to the supergravity equations for the supergravity multiplet and one tensor multiplet of $D = 6$, $N = (2, 0)$ supergravity expanded around the $AdS_3 \times \mathbf{S}^3$ metric and a self-dual three-form. We give the identification of the string vertex operator components in terms of the supergravity fields.

In flat space, the vertex operator field components can be described by representations of the $D = 6$ little group SO(4). The supergravity multiplet is $(3, 3) + 5(3, 1) + 4(3, 2)$ and the tensor multiplet is $(1, 3) + 5(1, 1) + 4(1, 2)$. Here $(3, 3)$ is the graviton field g_{mn}, the oscillation around the flat metric; the two-form b_{mn} has SO(4) spin content $(1, 3) + (3, 1)$; and the dilaton ϕ is $(1, 1)$. The bispinor fields F^{++ab}, V_{ab}^{--}, A_b^{+-a}, A_a^{-+b} represent four self-dual tensors $(3, 1)$ and four scalars $(1, 1)$. The four gravitinos $4(3, 2)$ and four spinors $4(1, 2)$ are labelled by χ_m^{+a}, $\overline{\chi}_m^{+\overline{a}}$, ξ_{ma}^{-}, $\overline{\xi}_{m\overline{a}}^{-}$.

In $AdS_3 \times S^3$ space, the number of physical degrees of freedom in the vertex operator (2.13) remains the same as in the flat case, but the field g_{mn} is now related to the oscillation around the $AdS_3 \times S^3$ metric. For this metric to be a solution of the supergravity equations [5], there must be also either a non-vanishing (real) self-dual tensor field or a non-vanishing (real) anti-self-dual tensor field. Expanding around this classical solution, we will find that we need to choose one of the real *self-dual* tensor fields to be non-vanishing at zeroeth order, to be able to identify these linearized supergravity equations with the constraints of the string model.

We will see that the two-form b_{mn} is a linear combination of *all* the oscillations corresponding to the five self-dual tensor fields and the anti-self-dual tensor field, including the oscillation with non-vanishing background. In flat space, b_{mn} corresponds to a state in the Neveu-Schwarz sector. In our curved space case, the string model describes vertex operators for AdS_3 background with Ramond-Ramond flux. When matching the vertex operator component fields with the supergravity oscillations, we find that not only the bispinor V_{ab}^{--} (which is a Ramond-Ramond field in the flat space case), but also the tensor b_{mn} include supergravity oscillations with non-vanishing self-dual background. We use the Romans' variables [5] to describe the AdS supergravity equations, where the three-form field strengths are labelled by one anti-self-dual K_{mnp} and five self-dual H^i_{mnp} fields. Our procedure will be to start in this section with the supergravity equations, and rewrite them as equations for the field combinations that occur in the string theory vertex operators. We then show that these equations are equivalent to the string constraint equations (4.3)–(4.8). We list the field identifications at the end of this section.

In the bosonic sector, the supergravity equations are

$$
\begin{aligned}
D^p D_p \phi^i &= \frac{2}{3}\, H^i_{mnp}\, K^{mnp} \\
H^i_{mnp} &= \frac{1}{6}\, e_{mnp}{}^{qrs}\, H^i_{qrs}\,, \\
K_{mnp} &= -\frac{1}{6}\, e_{mnp}{}^{qrs}\, K_{qrs} \\
R_{mn} &= -H^i_{mpq}\, H_n^{ipq} - K_{mpq}\, K_n^{pq} - D_m\phi^i\, D_n\phi^i\,,
\end{aligned} \tag{5.1}
$$

where $e_{mnpqrs} \equiv \frac{1}{\sqrt{-\tilde{g}}}\,\tilde{g}_{mm'}\cdots\tilde{g}_{ss'}\epsilon^{m'n'p'q'r's'}$ for a metric $\tilde{g}$, and $\epsilon^{123456}=1$, $1\le i\le 5$.

The string vertex operator components are related to the supergravity fluctuations around the $AdS_3\times S^3$ metric $\overline{g}_{mn}$ and the three-form field strength $\overline{H}^i_{mpq} = \delta^{i1}\overline{H}^1_{mpq}$, where $\overline{H}^1_{mpq} = \frac{1}{2}\,(\sigma_m\sigma_p\sigma_q)_{ab}\delta^{ab}$, and $\overline{g}_{mn}$ is given in (4.17). As in [6], we can parametrize these fluctuations as

$$
\tilde{g}_{mn} = \overline{g}_{mn} + h_{mn}\,, \qquad H^i_{mpq} = \overline{H}^i_{mpq} + g^i_{mpq}\,, \qquad K_{mpq} = g^6_{mpq} + \overline{H}^j_{mpq}\phi^j\,, \tag{5.2}
$$

where we label the scalar fluctuations by ϕ^i since they have vanishing background. The anti-self-dual tensor K_{mpq} also has vanishing background, but is parametrized as above so that the fluctuations g^i_{mpq} and g^6_{mpq} are exact three-forms [5].

The linearized supergravity equations are given by

$$
D^p D_p \phi^i = \frac{2}{3}\,\overline{H}^i_{prs}\, g^{6prs} \tag{5.3}
$$

and

$$
\begin{aligned}
&\frac{1}{2}D^p D_p\, h_{rs} - \overline{R}_{\tau rs\lambda}\, h^{\tau\lambda} + \frac{1}{2}\overline{R}_r^\tau\, h_{\tau s} + \frac{1}{2}\overline{R}_s^\tau\, h_{\tau r} - \\
&\quad -\frac{1}{2}D_s D^p h_{pr} - \frac{1}{2}D_s D^p h_{pr} + \frac{1}{2}D_r D_s h^p_p = -\overline{H}_r^{i\ pq}\, g^i_{spq} - \overline{H}_s^{i\ pq}\, g^i_{rpq} + \\
&\qquad\qquad + 2h^{pt}\,\overline{H}^{i\ q}_{rp}\,\overline{H}^i_{stq}
\end{aligned} \tag{5.4}
$$

and

$$
\begin{aligned}
D^p H_{prs} &= -2\,\overline{H}^i_{prs}\, D^p\phi^i + \\
&\quad + B^i\left[-\overline{H}_r^{i\ pq} D_p\, h_{qs} + \overline{H}_s^{i\ pq} D_p\, h_{qr} + \overline{H}^{i\ q}_{rs} D^p h_{pq} - \frac{1}{2}\overline{H}^{i\ q}_{rs} D_q h^p{}_p\right]
\end{aligned} \tag{5.5}
$$

where we have defined $H_{prs} \equiv g^6_{prs} + B^i\, g^i_{prs}$ as a combination of the supergravity exact forms $g^6 \equiv db^6$, $g^i \equiv db^i$, since we will equate this with the string field strength $H = db$. For the moment we let B^i be arbitrary constants. In zeroeth order, the equations are $\overline{R}_{rs} = -\overline{H}^i_{rpq}\,\overline{H}_s^{i\ pq}$.

Eq. (5.5) follows from the first order linearized duality equations

$$
\begin{aligned}
\frac{1}{6}\overline{e}_{mnp}{}^{qrs}\, g^6_{qrs} &= -g^6_{mnp} - 2\phi^i\overline{H}^i_{mnp} \\
\frac{1}{6}\overline{e}_{mnp}{}^{qrs}\, g^i_{qrs} &= g^i_{mnp} + \overline{H}^i_{mpq}\, h^q_n - \overline{H}^i_{npq}\, h^q_m - \overline{H}^i_{mnq}\, h^q_p + \frac{1}{2}\overline{H}^i_{mnp} h^q_q
\end{aligned} \tag{5.6}
$$

with $\overline{e}_{mnpqrs} \equiv \frac{1}{\sqrt{-\overline{g}}}\,\overline{g}_{mm'}\ldots\overline{g}_{ss'}\epsilon^{m'n'p'q'r's'}$. We note that the fluctuations g^6_{mnp}, g^i_{mnp} although exact, do not have definite duality due the presence of the metric.

To equate the supergravity equations (5.3)–(5.5) with the string constraints of section 4, we remember that the constraints hold for the string field combination $G_{mn} \equiv g_{mn} + b_{mn} + \overline{g}_{mn}\phi$. To put (5.5) and (5.4) in a similar form, and using $H_{prs} \equiv \partial_p b_{rs} + \partial_r b_{sp} + \partial_s b_{pr}$, we rewrite them as

$$\begin{aligned}
\frac{1}{2}D^pD_pb_{rs} = &-\overline{H}^1_{prs}\, D^p\phi^1 - \overline{H}^{1\ pq}_{r}\left[\frac{1}{2}B^1\, D_p\, h_{qs}\right] + \overline{H}^{1\ pq}_{s}\left[\frac{1}{2}B^1\, D_p\, h_{qr}\right] + \\
&+\frac{1}{2}B^1\left[\overline{H}^{1\ q}_{rs}\, D^p h_{pq} - \frac{1}{2}\overline{H}^{1\ q}_{rs} D_q h^p{}_p\right] + \overline{R}_{\tau r s\lambda} b^{\tau\lambda} - \frac{1}{2}\overline{R}^{\ \tau}_{r}\, b_{\tau s} - \frac{1}{2}\overline{R}^{\ \tau}_{s}\, b_{r\tau} + \\
&+\frac{1}{2}D_rD^pb_{ps} + \frac{1}{2}D_sD^pb_{rp}
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
\frac{1}{2}D^pD_p\, h_{rs} = &-\overline{H}^{1\ pq}_{r}\left[\frac{2}{B^1}\, D_p b_{qs}\right] + \overline{H}^{1\ pq}_{s}\left[\frac{2}{B^1}\, D_p b_{rq}\right] + \\
&-\overline{H}^{1\ pq}_{r}\,\frac{1}{B^1}\left[-g^6_{spq} - B^I g^I_{spq}\right] - \overline{H}^{1\ pq}_{s}\,\frac{1}{B^1}\left[-g^6_{rpq} - B^I g^I_{rpq}\right] + \\
&+2h^{pt}\,\overline{H}^{1\ q}_{rp}\,\overline{H}^1_{stq} + \overline{R}_{\tau r s\lambda}\, h^{\tau\lambda} - \frac{1}{2}\overline{R}^{\ \tau}_{r}\, h_{\tau s} - \frac{1}{2}\overline{R}^{\ \tau}_{s}\, h_{r\tau} + \\
&+\frac{1}{2}D_rD^ph_{ps} + \frac{1}{2}D_sD^ph_{rp} - \frac{1}{2}D_sD_rh^p_p - \\
&-D_r\overline{H}^{1\ pq}_{s}\,\frac{1}{B^1}b_{pq} - D_s\overline{H}^{1\ pq}_{r}\,\frac{1}{B^1}b_{pq}\,, \qquad \text{where } 2 \le I \le 5.
\end{aligned} \tag{5.8}$$

The string graviton field g_{rs} is chosen to be traceless and can be written in terms of the supergravity fields as

$$g_{rs} = h_{rs} - \frac{1}{6}\overline{g}_{rs}\, h^\lambda{}_\lambda\,. \tag{5.9}$$

Using (5.9) and the gauge conditions (4.14), we find (5.7) and (5.8) to be

$$\begin{aligned}
\frac{1}{2}D^pD_pb_{rs} = &-\overline{H}^{1\ pq}_{r}\, D_p\left[g_{qs} + \overline{g}_{qs}\phi\right] + \overline{H}^{1\ pq}_{s}\, D_p\left[g_{qr} + \overline{g}_{qr}\phi\right] - \\
&-\overline{R}_{\tau r s\lambda}\, b^{\tau\lambda} - \frac{1}{2}\overline{R}^{\ \tau}_{r}\, b_{\tau s} - \frac{1}{2}\overline{R}^{\ \tau}_{s}\, b_{r\tau} + \\
&+\overline{H}^1_{prs}\, D^p\left[-\phi^1 - 3\,\phi\right],
\end{aligned} \tag{5.10}$$

$$\begin{aligned}
\frac{1}{2}D^pD_p\left(g_{rs} + \overline{g}_{rs}\phi\right) = &-\overline{H}^{1\ pq}_{r}\, D_pb_{qs} + \overline{H}^{1\ pq}_{s}\, D_pb_{rq} + 2(g^{pt} + \overline{g}^{pt}\phi)\,\overline{H}^{1\ q}_{rp}\,\overline{H}^1_{stq} + \\
&+\overline{R}_{\tau r s\lambda}\left(g^{\tau\lambda} + \overline{g}^{\tau\lambda}\phi\right) - \frac{1}{2}\overline{R}^{\ \tau}_{r}\left(g_{\tau s} + \overline{g}_{\tau s}\phi\right) - \\
&-\frac{1}{2}\overline{R}^{\ \tau}_{s}\left(g_{r\tau} + \overline{g}_{r\tau}\phi\right) - D_rD_s\phi - \frac{1}{3}D_rD_sh^\lambda{}_\lambda + \\
&+\overline{R}_{rs}\left[-\frac{1}{3}\, h^\lambda{}_\lambda + \phi^1 + 2\phi\right] - \\
&-\overline{g}_{rs}\, D^pD_p\left[\frac{1}{12}\, h^\lambda{}_\lambda - \frac{1}{4}\,\phi^1 - \frac{1}{2}\phi\right] + \\
&+\overline{H}^{1\ pq}_{r}\,\frac{1}{2}[B^Ig^I_{spq}] + \overline{H}^{1\ pq}_{s}\,\frac{1}{2}[B^Ig^I_{rpq}]\,,
\end{aligned} \tag{5.11}$$

where we have used (5.3) and let $B^1 = 2$. Note that $-\frac{1}{2}\,\overline{H}^{1\ q}_{rs}(\sigma_m\sigma^q\sigma_n)_{ab}\delta^{ab}\,b_{mn} = -2\overline{R}_{\tau rs\lambda}\,b^{\tau\lambda}$. As discussed below, we make the field identifications:

$$\frac{1}{4}F^{++gh}_{\rm asy}\,\sigma^{ab}_r\sigma^{ef}_s\,\delta_{ah}\delta_{be}\delta_{gf} \equiv \overline{H}^1_{prs}\,D^p[-\phi^1 - 3\phi] \tag{5.12}$$

$$\begin{aligned}\frac{1}{4}F^{++gh}_{\rm sym}\,\sigma_{rga}\sigma_{shb}\,\delta^{ab} \equiv\ & \overline{H}^{1\ pq}_r\,\frac{1}{2}[B^I g^I_{spq}] + \overline{H}^{1\ pq}_s\,\frac{1}{2}[B^I g^I_{rpq}] - \\ & - D_rD_s\phi - \frac{1}{3}D_rD_s h^\lambda_{\ \lambda} + \overline{R}_{rs}\left[-\frac{1}{3}\,h^\lambda_{\ \lambda} + \phi^1 + 2\phi\right] - \\ & - \overline{g}_{rs}\,D^pD_p\left[\frac{1}{12}\,h^\lambda_{\ \lambda} - \frac{1}{4}\,\phi^1 - \frac{1}{2}\phi\right].\end{aligned} \tag{5.13}$$

These identifications are appropriate for the following reasons. From the trace of (5.13), $0 = D^pD_p\,[\,2\phi - \frac{5}{6}\,h^\lambda_{\ \lambda} + \frac{3}{2}\phi^1\,]$, and the trace of (5.8)

$$D^pD_p\left[\frac{5}{6}\,h^\lambda_{\ \lambda} + \phi - \frac{1}{2}\phi^1\right] = 0\,, \tag{5.14}$$

we find

$$D^pD_p\,[-\phi^1 - 3\phi] = 0\,. \tag{5.15}$$

Eq. (5.15), derived here from the supergravity equations, is also the *string* equation (4.20) for $F^{++ab}_{\rm asy}$ derived in section 4 *when* $F^{++ab}_{\rm asy}$ *is defined by* (5.12).

To make contact with the string equation (4.21) for $F^{++ab}_{\rm sym}$, we multiply (5.13) by $\sigma^{s\ HB}\,D_r$, so that

$$\begin{aligned}\frac{1}{4}\left[\delta^{Ba}\,\sigma^r_{ga}\,D_r\,F^{++gH}_{\rm sym} - \delta^{Ha}\,\sigma^r_{ga}\,D_r\,F^{++gB}_{\rm sym}\right] &= -\frac{1}{2}\,\epsilon^{HB}{}_{cd}\,\sigma^{r\,cd}\,D_r[\phi^1 + 3\phi] \\ &= -\frac{1}{4}\,\epsilon^{BH}{}_{cd}\,F^{++cd}_{\rm asy}\,,\end{aligned} \tag{5.16}$$

where we have used the equations of motion $D^pg^I_{pqs} = 0$, $2 \le I \le 5$. From (5.3), the equations for the scalars are

$$\begin{aligned} D^pD_p\phi^I &= 0\,, \qquad 2 \le \mathrm{I} \le 5 \\ D^pD_p\,\phi^1 &= \frac{2}{3}\overline{H}^{1prs}\,H_{prs} + 2\overline{R}_{rp}\,g^{rp}\,.\end{aligned} \tag{5.17}$$

From (5.14) and (5.17) we make the field identification between the string dilaton field ϕ and the supergravity scalars $h^\lambda_{\ \lambda}, \phi^i$:

$$\phi \equiv \frac{1}{2}\,\phi^1 - \frac{5}{6}h^\lambda_{\ \lambda} + C^I\phi^I\,. \tag{5.18}$$

Note that for $AdS_3 \times S^3$ we have

$$2(g^{pt} + \overline{g}^{pt}\phi)\overline{H}^{1\ q}_{rp}\,\overline{H}^1_{stq} = -2\overline{R}_{\tau rs\lambda}(g^{\tau\lambda} + \overline{g}^{\tau\lambda}\phi)\,. \tag{5.19}$$

With these field identifications, we can combine (5.10) and (5.11) to find from the supergravity that

$$\begin{aligned}\frac{1}{2}D^pD_pG_{rs} = &-\overline{H}^{1\ pq}_r\, D_p\, G_{qs} + \overline{H}^{1\ pq}_s\, D_p\, G_{rq} - \\ &- \overline{R}_{\tau rs\lambda}\, G^{\tau\lambda} - \frac{1}{2}\overline{R}_r^{\ \tau}\, G_{\tau s} - \frac{1}{2}\overline{R}_s^{\ \tau}\, G_{r\tau} + \\ &+ \frac{1}{4}F^{++gh}_{\text{asy}}\, \sigma^{ab}_r \sigma^{ef}_s\, \delta_{ah}\delta_{be}\delta_{gf} + \\ &+ \frac{1}{4}F^{++gh}_{\text{sym}}\, \sigma_{rga}\sigma_{shb}\, \delta^{ab}\,,\end{aligned} \tag{5.20}$$

which is the zero laplacean string conditions (4.18), (4.19).

For the string constraints on V^{--}_{ab}, we have from the trace of (4.22)

$$8\,\overline{R}_{mn}\, g^{mn} = \frac{1}{2}\, D^pD_p\, \delta^{cd}\, V^{--}_{cd} + D^m\sigma_{mab}\, \delta^{ac}\delta^{bd}\, V^{--}_{cd}\,, \tag{5.21}$$

and acting on (4.22) with $(\sigma^{rcd}D_r - \delta^{cd})$ and using the supergravity equations (5.17) we find

$$-8\phi^1 + 8C^I_{--}\phi^I = \sigma^{ncd}\, D_n\, V^{--}_{cd} - 2\,\delta^{cd}\, V^{--}_{cd}\,. \tag{5.22}$$

From (4.22) and the second order equation for g^i_{qrs} which follows from (5.6), one can show that $(\,\sigma^{mac}\,\sigma^{nbd}\, D_m\, D_n\, V^{--}_{cd}\,)_{\text{sym}}$ involves a combination of the supergravity fields $(\sigma_p\sigma_m\sigma_n)^{ab}\, g^1_{pmn}$ and $(\sigma_p\sigma_m\sigma_n)^{ab}\, B^I_{--}g^I_{pmn}$.

To summarize the vertex operator components in terms of the supergravity fields $g^i_{prs}, g^6_{prs}, h_{rs}, \phi^i$, $1 \le i \le 5$, (and $2 \le I \le 5$ below):

$$\begin{aligned}H_{prs} &\equiv g^6_{prs} + 2\, g^1_{prs} + B^I\, g^I_{prs} && (5.23)\\ g_{rs} &\equiv h_{rs} - \frac{1}{6}\overline{g}_{rs}\, h^\lambda_{\ \lambda} && (5.24)\\ \phi &= -\frac{1}{3}\, h^\lambda_{\ \lambda} && (5.25)\\ F^{++ab}_{\text{sym}} &= \frac{2}{3}(\sigma_p\sigma_r\sigma_s)^{ab}\, B^I\, g^I_{prs} + \delta^{ab}\,\phi^{++} \\ F^{++ab}_{\text{asy}} &= \sigma^{p\,ab}\, D_p\,\phi^{++} \\ \phi^{++} &= 4C^I\,\phi^I\,, && (5.26)\end{aligned}$$

which follows from choosing the graviton trace $h^\lambda_{\ \lambda}$ to satisfy $\phi^1 - h^\lambda_{\ \lambda} \equiv -2\,C^I\phi^I$. This plays the role of the harmonic coordinate condition which occurs in flat space string amplitudes.

The combinations $C^I\phi^I$ and $B^I g^I_{prs}$ reflect the $\mathrm{SO}(4)_{\mathrm{R}}$ symmetry of the $D = 6, N = (2,0)$ theory on $AdS_3 \times S^3$. We relabel $C^I = C^I_{++}$, $B^I = B^I_{++}$. To define the remaining string components in terms of supergravity fields, we consider linearly independent quantities $C^I_\ell \phi^I$, $B^I_\ell g^I_{prs}$, $\ell = ++, +-, -+, --$.

$$\begin{aligned}
F_{\text{sym}}^{+-ab} &= \frac{2}{3}(\sigma_p\sigma_r\sigma_s)^{ab}\, B^I_{+-}\, g^I_{prs} + \delta^{ab}\,\phi^{+-} \\
F_{\text{asy}}^{+-ab} &= \sigma^{p\,ab}\, D_p\,\phi^{+-} \\
\phi^{+-} &= 4C^I_{+-}\,\phi^I \\
F_{\text{sym}}^{-+ab} &= \frac{2}{3}(\sigma_p\sigma_r\sigma_s)^{ab}\, B^I_{-+}\, g^I_{prs} + \delta^{ab}\,\phi^{-+} \\
F_{\text{asy}}^{-+ab} &= \sigma^{p\,ab}\, D_p\,\phi^{-+} \\
\phi^{-+} &= 4C^I_{-+}\,\phi^I
\end{aligned} \tag{5.27}$$

V_{ab}^{--} is given in terms of the fourth tensor/scalar pair $C^I_{--}\ \phi^I$, $B^I_{--} g^I_{mnp}$ through

$$D^p D_p\, V_{cd}^{--} - \delta^{gh}\sigma^p_{ch}\, D_p\, V_{gd}^{--} + \delta^{gh}\sigma^p_{dh}\, D_p\, V_{cg}^{--} + \frac{1}{2}\epsilon_{cd}{}^{gh}\, V_{gh}^{--} = -8\sigma^m_{ce}\,\sigma^n_{df}\,\delta^{ef}\, G_{mn}\,. \tag{5.28}$$

The fermion constraints in (4.24) imply the linearized AdS supergravity equations for the gravitinos and spinors, due to the above correspondence for the bosons and the supersymmetry of the two theories.

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A. Sigma matrices

The sigma matrices σ^{mab} [4, 7] satisfy the algebra

$$\sigma^{mab}\sigma^n_{ac} + \sigma^{nab}\sigma^m_{ac} = \eta^{mn}\delta^b_c \tag{A.1}$$

in flat space, where η^{mn} is the six-dimensional Minkowski metric, and $1 \le a \le 4$. Sigma matrices with lowered indices are defined by $\sigma^m_{ab} = \frac{1}{2}\epsilon_{abcd}\sigma^{mcd}$, although for other quantities indices are raised and lowered with δ^{ab}, so we distinguish σ^m_{ab} from $\delta_{ac}\,\delta_{bd}\,\sigma^{mcd}$. The following identities are useful

$$\sigma^{mab}\sigma^n_{cd}\,\eta_{mn} = \delta^a_c\delta^b_d - \delta^a_d\delta^b_c\,, \qquad \sigma^{mab}\sigma^{ncd}\,\eta_{mn} = \epsilon^{abcd}\,. \tag{A.2}$$

Also (A.1) and (A.2) hold in curved space with η_{mn} replaced by the $AdS_3 \times S^3$ metric $\overline{g}_{mn}$. Then $\overline{g}^{mn} = \frac{1}{2}\sigma^{mab}\sigma^n_{ab}$ as in (4.17).

The product of sigma matrices $\langle G_{mnp}\rangle \equiv (\sigma_m\sigma_n\sigma_p)_{ab}\,\delta^{ab}$ is self-dual. Similarly $(\sigma_m\sigma_n\sigma_p)^{ab}\,\delta_{ab}$ is anti-self-dual. The sigma matrices σ^{mab} describe the coupling of two **4**'s to a **6** of SU(4). They are SU(4) singlets, whereas δ^{ab}, though an SO(4) singlet, transforms in the symmetric second rank tensor **10** of SU(4). From the group theory properties of the sigma matrices, we see that $\langle G_{mnp}\rangle$ is a **10** of SU(4), so the product of the two self-dual tensors, which is a singlet of SU(4), must be zero $\langle G^{mnp}\rangle\langle G_{mnp}\rangle = 0$, since $\mathbf{10}\times\mathbf{10} = \mathbf{20}' + \mathbf{35} + \mathbf{45}$ does not contain a singlet. This is consistent with the general identities [5] that

$$X_{[m}{}^{np}Y_{r]\,np} = 0 = X^{mnp}\,Y_{mnp}\,, \tag{A.3}$$

when the tensors X, Y are both self-dual (or both anti-self-dual); and

$$X_{(m}{}^{np}Z_{r)\,np} = \frac{1}{6}g_{mr}\,X^{nps}\,Z_{nps}\,, \tag{A.4}$$

for X, Z of opposite duality.

B. Covariant derivatives on a group manifold

On a group manifold, there is always a right-invariant vielbein and a left-invariant vielbein; for the SO(4) case, we write them as

$$e^{mab} = -\sigma^{mab} \tag{B.1}$$

and

$$e^{m\bar{a}\bar{b}} = \sigma^{m\bar{a}\bar{b}} \tag{B.2}$$

respectively. The vielbein obeys

$$e^{mab}\,e_m^{cd} = \epsilon^{abcd}\,, \qquad e^{mab}\,e^{ncd}\,\epsilon_{abcd} = 4\,\overline{g}^{mn}\,. \tag{B.3}$$

Using the right-invariant vielbein, the covariant derivative on the group manifold can be conveniently written

$$D_mV_{gh} = \partial_mV_{gh} - \frac{1}{4}\,f_{gh}{}^{cd}{}_{ef}\,e_m^{ef}\,V_{cd} \tag{B.4}$$

for V_{gh} antisymmetric in g, h. Here by f we mean the structure constants f^A_{BC}, with A, B, and C being Lie algebra indices, but for the group SO(4), the Lie algebra is the second rank antisymmetric tensor representation, and we write a Lie algebra index as a pair ab with antisymmetry understood. It is convenient to refer the covariant derivative to the right-invariant vielbein, defining $\mathcal{T}_L^{ab} = e^{mab}D_m$. Similarly, we define $\mathcal{T}_R^{\bar{a}\bar{b}} = e^{m\bar{a}\bar{b}}D_m$, using the left-invariant vielbein.

It is also convenient to let t_L be a covariant derivative that is defined by using the right-invariant vielbein and setting the spin connection to zero; likewise, t_R is a covariant derivative defined using the left-invariant vielbein and setting the spin connection to zero. Thus, t_L and t_R are the covariant derivatives defined using the right- and left-invariant framings of the group manifold.

From the above formulas, one can deduce the relation between t_L and $\mathcal{T}_L$. On a Lie algebra valued field V_{gh} $(= -V_{hg})$ we have

$$t_L^{ab} V_{gh} = \mathcal{T}_L^{ab} V_{gh} + \frac{1}{2} f_{gh}{}^{cdab} V_{cd} \,. \tag{B.5}$$

From this, we can deduce that on a spinor field the relation is

$$t_L^{ab} V_g = \mathcal{T}_L^{ab} V_g + \frac{1}{2} \delta_g^a \, \delta^{bc} V_c - \frac{1}{2} \delta_g^b \, \delta^{ac} V_c \tag{B.6}$$

and on a field with two pairs of anti-symmetric indices it is

$$t_L^{ab} V_{gh\,jk} = \mathcal{T}_L^{ab} V_{gh\,jk} + \frac{1}{2} f_{gh}{}^{cdab} V_{cd\,jk} + \frac{1}{2} f_{jk}{}^{cdab} V_{gh\,cd} \,. \tag{B.7}$$

The above formulas have obvious counterparts for t_R and $\mathcal{T}_R$.

It is convenient, as in the case of the sigma matrices, to raise and lower the index pairs on the structure constants with ϵ_{abcd}, so $\frac{1}{2}\epsilon_{abef} f_{gh}{}^{cdab} = f_{gh}{}^{cd}{}_{ef}$. Otherwise indices are raised and lowered with δ^{ab}.

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